Patterned Erasure Correcting Codes for Low Storage-Overhead Blockchain Systems

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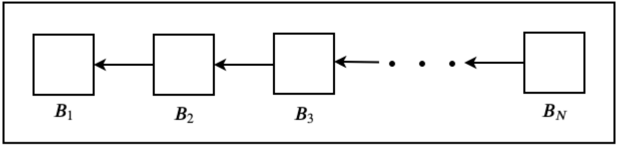
Blockchain Ledger



- N blocks $B_1, B_2, \ldots, B_N$.
- Stored in the form of a hash chain $\implies$ Tamper proof

Storage Burden in Blockchains

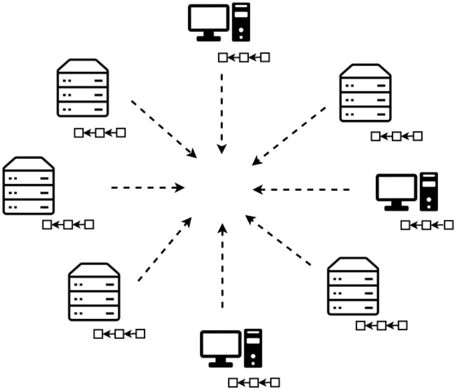


Figure: P2P Network

- P2P Network of n nodes

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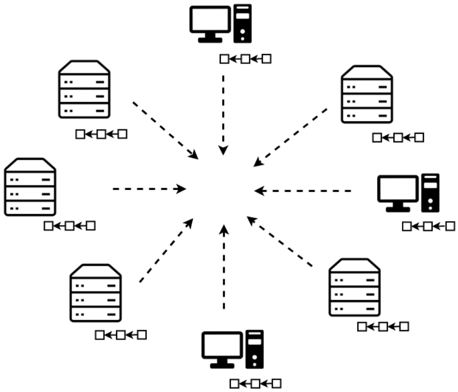

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- Each node stores full ledger
- Decentralized

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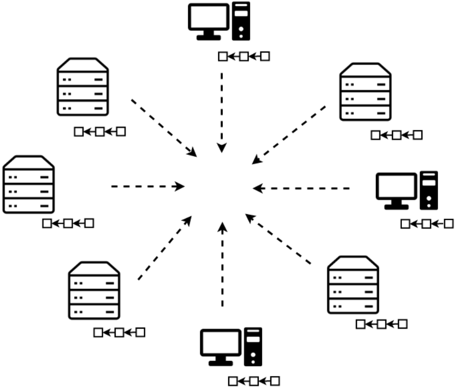

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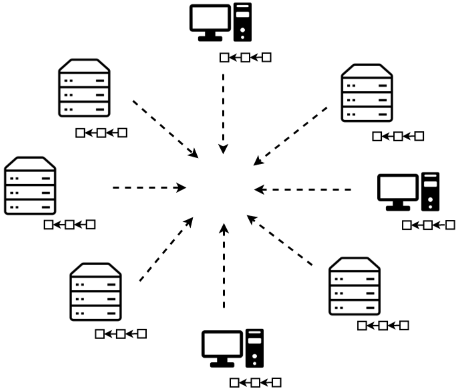

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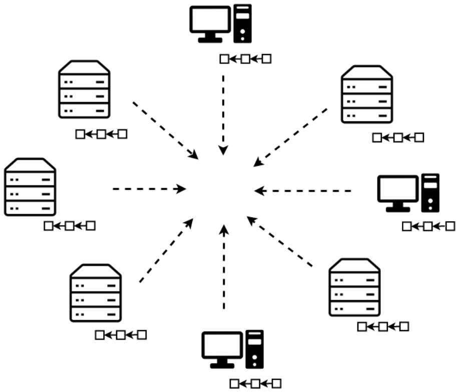

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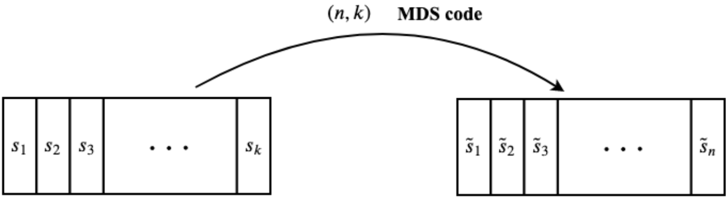
Significant storage cost

Goal: Reduce storage costs without reducing blockchain availability

Prior Work: Coded Sharding[1,2]



- Blockchain of size B partitioned into k shards $s_1, s_2, \ldots, s_k$
- n coded shards $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n$ generated using (n, k) MDS code

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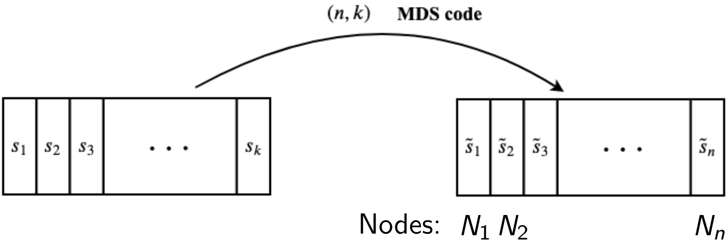

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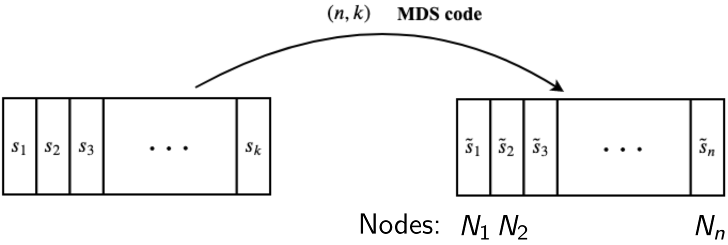

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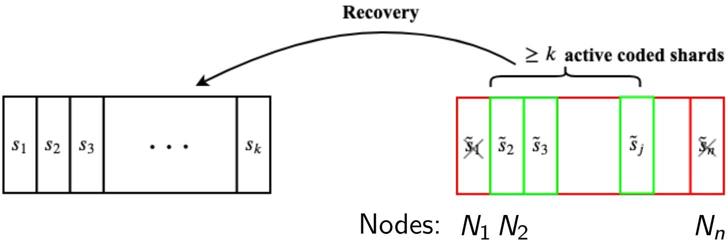

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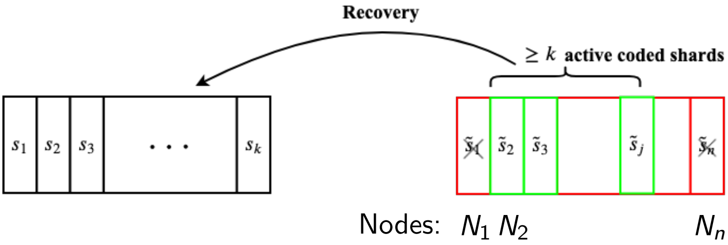

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Patterned Erasures

- In practice, nodes fail periodically[3]
- Different nodes with different periodicities

$\implies$ only specific patterns of erasures possible

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- Can reduce storage per node by designing codes which correct only these erasure patterns

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Goal: Storage Reduction in Patterned Erasure Model

- For a blockchain of size B with n nodes $\{N_1, N_2, \ldots N_n\}$ and erasure patterned set $\mathcal{P} = \{P_1, P_2, \ldots, P_{|\mathcal{P}|}\}$, design a code which guarantees to corrects all erasure patterns in $\mathcal{P}$ and has minimum average storage per node

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We can get a better average storage per node by relaxing this condition

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Condition for Blockchain recoverability:

- For each patterned erasure set, the number of coded shards in its complement should be at least k

Optimal Shard Allocation

- Considering k and x_i's as variables
- Code construction involves solving the following:
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Integer Optimization

$$\min_{x_1,\ldots,x_n,k} \quad \frac{B}{n}\frac{\sum_{i=1}^{n} x_i}{k}$$

$$s.t \sum_{i:N_i \in \bar{P}_j} x_i \geq k, j = 1, 2, \ldots, |\mathcal{P}|$$

$$x_i \in \mathrm{Z}^+, i = 1, 2, \ldots, n$$

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- Optimal solution $(\mathbf{x}^*, k^*)$.

Code Construction

PARE (Pattern Aware Redundancy for Erasures)- Code :

- m^{th} coded shard at N_i: $\alpha_{i,m}^{1} s_1 + \alpha_{i,m}^{2} s_2 + \ldots + \alpha_{i,m}^{k^*} s_{k^*}$, $1 \leq m \leq x_i^*$

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$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \alpha^1_{i,1} & \alpha^2_{i,1} & \alpha^3_{i,1} & \ldots & \ldots & \alpha^{k^*}_{i,1} \\ \alpha^1_{i,2} & \alpha^2_{i,2} & \alpha^3_{i,2} & \ldots & \ldots & \alpha^{k^*}_{i,2} \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \alpha^1_{i,x^*_i} & \alpha^2_{i,x^*_i} & \alpha^3_{i,x^*_i} & \ldots & \ldots & \alpha^{k^*}_{i,x^*_i} \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \end{bmatrix}$$

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- Can always choose a sufficiently large field to get required $\alpha^{\nu}_{i,m}$

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- Can always choose a sufficiently large field to get required $\alpha^{\nu}_{i,m}$
- E.g. can choose $\alpha^{\nu}_{i,m}$ to form Vandermonde type matrices

Equivalence with Linear Programming

Integer Optimization

$$\min_{x_1,\ldots,x_n,k} \quad \frac{B}{n}\frac{\sum_{i=1}^{n} x_i}{k}$$

$$s.t \sum_{i:N_i \in \bar{P}_j} x_i \geq k, j = 1, 2, \ldots, |\mathcal{P}|$$

$$x_i \in \mathrm{Z}^{+}, i = 1, 2, \ldots, n; \; k \in \mathrm{Z}^{++}$$

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Linear Program

$$\min_{y_1,\ldots,y_n} \sum_{i=1}^{n} y_i$$

$$s.t \sum_{i:N_i\in\bar{P}_j} y_i \geq 1, j = 1, 2, \ldots, |\mathcal{P}|$$

$$y_i \geq 0, i = 1, 2, \ldots, n$$

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Equivalence:

- If $\mathbf{y}^* = (y_1^*, y_2^*, \ldots, y_n^*)$ is an optimal solution of the LP, then choose k^* st. $k^* \times \mathbf{y}^* = (k^* y_1^*, k^* y_2^*, \ldots, k^* y_n^*)$ is integral and $\mathbf{x}^* = k^* \times \mathbf{y}^*$
- $(\mathbf{x}^*, k^*)$ is optimal for Integer Optimization problem

PARE-Example

- 6 Nodes: $\{N_1, N_2, N_3, N_4, N_5, N_6\}$

$$\mathcal{P} = \left\{ \begin{array}{c} \{N_1, N_3, N_4, N_5\} \\ \{N_1, N_3, N_6\} \\ \{N_2, N_3, N_5, N_6\} \\ \{N_1, N_2, N_4\} \\ \{N_4, N_6\} \end{array} \right\}$$

PARE-Example

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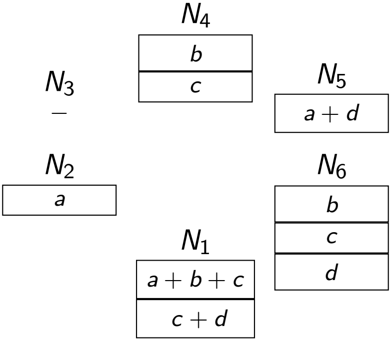

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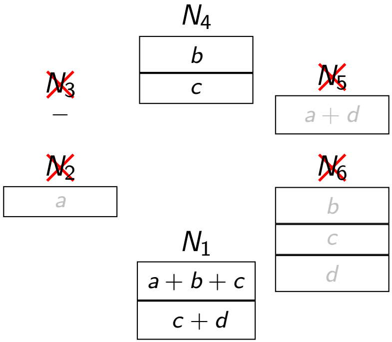

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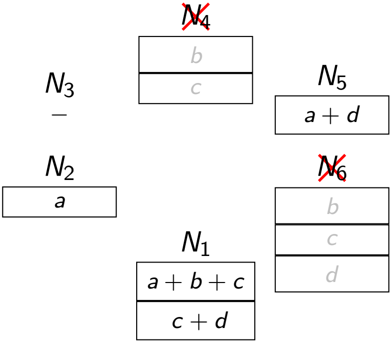

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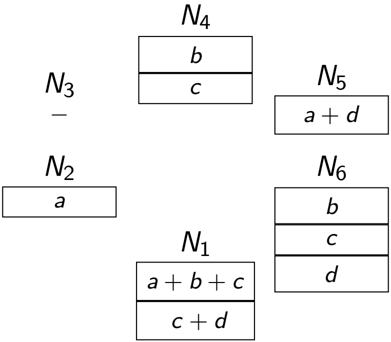

- Average storage per node using PARE-Code: 0.375B
- Average storage per node using (6,2) MDS code: 0.5B

Theoretical Analysis

Lemma

Average storage per node for PARE-Code is no more than $\frac{B}{n-t}$*, where* $t = \max|P_j|$*.*

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Proof Idea

For any coding scheme, if $B_1, B_2, \ldots, B_n$ are the amounts of the blockchain stored at $N_1, N_2, \ldots, N_n$ respectively, then B_i's must satisfy $\sum_{i:N_i \in \bar{P}_j} B_i \geq B \implies \frac{B_i}{B}$ is feasible in the LP.

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Proof Idea

For sufficiently large n, the probability that the $(n+1)^{st}$ *node has the same periodicity pattern as one of earlier nodes tends to 1.*

Condition for Redesign

LP for $(n+1)$ system:

$$\min_{y_1,\ldots,y_{n+1}} \mathbf{1}^T\mathbf{y} \qquad \max -\mathbf{b}^T\boldsymbol{\lambda}$$

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- In practice, we check this condition to decide if redesign is needed or not. With Probability 1 it is not needed.

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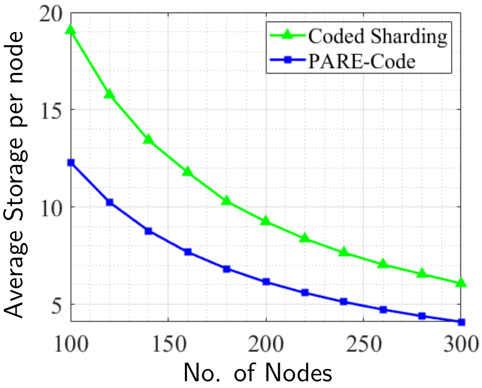
Average Storage



- Used $U = [5, 6, 7]$, $D = [1, 3, 5]$ and $B = 1024$

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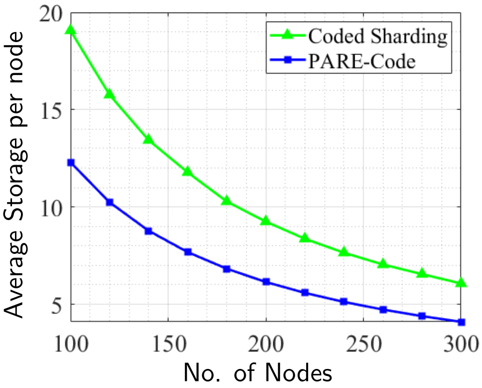
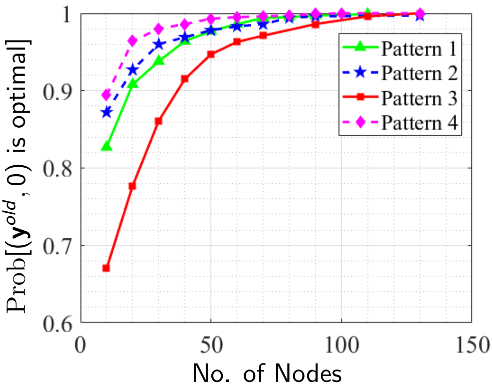


- Used $U = [5, 6, 7]$, $D = [1, 3, 5]$ and $B = 1024$
- PARE-Code has a lower average storage per node compared to coded sharding

Probability of Redesign



Pattern	U	D
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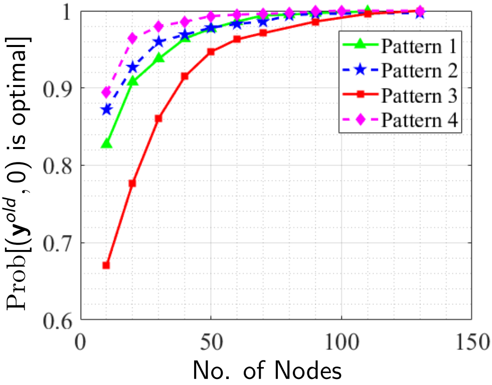

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- $\text{Prob}[(\mathbf{y}^{old}, 0)$ is optimal$] \rightarrow 1$ as number of nodes increases

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Conclusion and Ongoing Work

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- We provide a coding scheme which minimally corrects a predefined set of erasure patterns and is optimal in terms of average storage per node
- We prove that with high probability no redesign is needed using our code when there are sufficiently large number of nodes in the system

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Ongoing Work:

- Effect of node leaving the system
- Communication cost during recovery from erasures

Thank you!